Topological Aspects of Parasupersymmetry

Ali Mostafazadeh

Department of Physics, Sharif University of Technology
P. O. Box 11365-9161, Tehran, Iran, and
Institute for Studies in Theoretical Physics and Mathematics
P. O. Box 19395-1795, Tehran, Iran.

February 1, 2008

Abstract

Parasupersymmetric quantum mechanics is exploited to introduce a topological invariant associated with a pair of parameter dependent Fredholm (respectively elliptic differential) operators satisfying two compatibility conditions. An explicit algebraic expression for this topological invariant is provided. The latter identifies the parasupersymmetric topological invariant with the sum of the analytic (Atiyah-Singer) indices of the corresponding operators.

1 Introduction

Perhaps one of the most intriguing aspects of supersymmetry is its relation with the Atiyah-Singer index theorem [1]. It was Witten [4] who first recognized this relation in the context of supersymmetric quantum mechanics (SQM). The subsequent developments in this direction have led to supersymmetric proofs of the index theorem [5].

During the same period of development of supersymmetric proofs of the index theorem, i.e., mid 1980's, Rubakov and Spiridonov (R-S) [7] introduced their ($p =$

2)-parasupersymmetric quantum mechanics (PSQM). This involved a generalization of the superalgebra of SQM (see Eq. (6) below), namely the *parasuperalgebra*:

$$\mathcal{Q}^3 \;=\; 0\;, \quad [\mathcal{Q}, H] \;=\; 0\;, \tag{1}$$

$$\{\mathcal{Q}^2, \mathcal{Q}^\dagger\} + \mathcal{Q}\mathcal{Q}^\dagger\mathcal{Q} \;=\; 4\mathcal{Q}H\;. \tag{2}$$

The defining parasuperalgebra (1), (2) have since been generalized to arbitrary order $p > 2$, by Khare [8], and modified by Beckers and Debergh (B-D) [9]. B-D ($p = 2$)-parasuperalgebra is given by Eqs. (1) and

$$\left[\mathcal{Q}, \left[\mathcal{Q}^\dagger, \mathcal{Q}\right]\right] = 2\mathcal{Q}H\;. \tag{3}$$

In a preceding article [10], it is shown that a careful analysis of the defining parasuperalgebra (for both R-S and B-D types) provides important information on the degeneracy structure of the spectrum of the corresponding systems. In particular, postulating the existence of a parasupersymmetry involution (chirality) operator and supplementing (either of) the parasuperalgebra(s) with an additional relation expressing the Hamiltonian in terms of the parasupercharges, namely

$$H = \frac{1}{2}\left[(\mathcal{Q}\mathcal{Q}^\dagger)^2 + (\mathcal{Q}^\dagger\mathcal{Q})^2 - \frac{1}{2}(\mathcal{Q}\mathcal{Q}^{\dagger 2}\mathcal{Q} + \mathcal{Q}^\dagger\mathcal{Q}^2\mathcal{Q}^\dagger)\right]^{\frac{1}{2}}\;, \tag{4}$$

one can show that the integer

$$\begin{aligned}
\Delta^{(p=2)} &:= n^{\pi B} - 2n^{\pi F} \;=\; n_0^{\pi B} - 2n_0^{\pi F}\;, \qquad (5)\\
n^{\pi B} &:= \text{number of parabosonic states}\\
n^{\pi F} &:= \text{number of parafermionic states}\\
n_0^{\pi B} &:= \text{number of zero energy parabosonic states}\\
n_0^{\pi F} &:= \text{number of zero energy parafermionic states}
\end{aligned}$$

is a topological invariant. Furthermore, it is shown in [10] that $\Delta^{(p=2)}$ is a measure of parasupersymmetry breaking, i.e., the condition $\Delta^{(p=2)} \neq 0$ implies the exactness of parasupersymmetry. In this respect, it is quite similar to the Witten index of supersymmetry.

The purpose of the present letter is to explore the mathematical meaning of $\Delta^{(p=2)}$. In section 2, a brief discussion of SQM is presented to demonstrate the motivation for the proceeding analysis of PSQM. In section 3, a derivation of the expression for $\Delta^{(p=2)}$ is offered and the main result of the letter is presented. Section 4 includes the concluding remarks.

2 SQM and the Index Theorem

The main ingredient of SQM which makes its relation with the index theory possible, is its simple degeneracy structure. More precisely, the degeneracy structure of the spectrum of any supersymmetric quantum mechanical system is determined using only the defining superalgebra:

$$\mathcal{Q}^2 = 0,\ [\mathcal{Q}, H] \ =\ 0,\ \{\mathcal{Q}, \mathcal{Q}^\dagger\} \ =\ 2H\,, \tag{6}$$

and the properties of the supersymmetry involution (chirality) operator τ:

$$\tau^2 = 1,\ \tau^\dagger = \tau,\ \{\mathcal{Q}, \tau\} = 0\,. \tag{7}$$

In Eqs. (6) and (7), $\mathcal{Q}$ stands for (one of) the generator(s) of supersymmetry, $\mathcal{Q}^\dagger$ is its adjoint, and H is the Hamiltonian. The chirality operator τ induces a double grading of the Hilbert space, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\mathcal{H}_\pm := \{\psi \in \mathcal{H}\ :\ \tau\psi = \pm\psi\}\ . \tag{8}$$

The superalgebra (6) can be employed to show that the energy spectrum is non-negative and that each positive energy state of definite chirality is accompanied with another state of the same energy and opposite chirality, [4, 10]. In this sense, one says that the positive energy levels are *doubly degenerate*.

Introducing the self-adjoint generators:

$$Q_1 = \frac{1}{\sqrt{2}}(\mathcal{Q} + \mathcal{Q}^\dagger)\ , \qquad Q_2 = \frac{-i}{\sqrt{2}}(\mathcal{Q} - \mathcal{Q}^\dagger)\ , \tag{9}$$

one rewrites the superalgebra (6) in the form:

$$\begin{aligned}
\{Q_1, Q_2\} &= 0\,, && (10)\\
Q_1^2 \ =\ Q_2^2 &= H\,, && (11)\\
[Q_1, H] &= 0\,, && (12)\\
[Q_2, H] &= 0\,, && (13)\\
\{Q_1, \tau\} &= 0\,, && (14)\\
\{Q_2, \tau\} &= 0\,, && (15)\\
\tau^2 \ =\ 1 &\ ,\quad \tau^\dagger \ =\ \tau\,. && (16)
\end{aligned}$$

In view of (12), one can use the eigenvalues E and $q_1 = \pm\sqrt{E}$ of H and Q_1, to label the states. Here we choose not to include any other quantum numbers. Their presence will not interfere with the arguments presented in this letter.

For each positive energy level ($E > 0$),the $\{|E, \pm\sqrt{E}\rangle\}$ basis may be used to yield matrix representations of the relevant operators [10]. Let us denote by $\mathcal{H}_E$ the eigenspace associated with the eigenvalue E, then

$$Q_1|_{\mathcal{H}_E} = \sqrt{E}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sqrt{E}\sigma_3 \, , \qquad Q_2|_{\mathcal{H}_E} = \sqrt{E}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sqrt{E}\sigma_1,$$
$$\tau|_{\mathcal{H}_E} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \qquad H|_{\mathcal{H}_E} = E\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, ,$$

where σ_i, $i = 1, 2, 3$, are Pauli matrices. Let us transform into a basis where τ and H are diagonal. In such a basis:

$$\begin{aligned} Q_1|_{\mathcal{H}_E} &= \sqrt{E}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sqrt{E}\sigma_2 \, , \\ Q_2|_{\mathcal{H}_E} &= \sqrt{E}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sqrt{E}\sigma_1, \\ \tau|_{\mathcal{H}_E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= \sigma_3, \qquad H|_{\mathcal{H}_E} = E\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, . \end{aligned} \tag{17}$$

The fact that $trace(\tau|_{\mathcal{H}_E}) = 0$ is the very reason for the topological invariance of the Witten index [4]:

$$\begin{aligned} \text{index}_W &:= \text{trace}(\tau) \;=\; n^B - n^F \;=\; n_0^B - n_0^F \, , \qquad (18) \\ n^B &:= \text{number of bosonic states} \\ n^F &:= \text{number of fermionic states} \\ n_0^B &:= \text{number of zero energy bosonic states} \\ n_0^F &:= \text{number of zero energy fermionic states} \end{aligned}$$

Eq. (17) serves as a motivation for relating the Witten index with the analytic indices of Fredholm operators. To demonstrate this relationship, first one introduces the representation

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{pmatrix} \tag{19}$$

of the Hilbert space in which τ is (block-)diagonal. To obtain the representations of Q_i ($i = 1, 2$), one appeals to Eqs. (14) and (15). These together with (17) suggest:

$$Q_1 = \begin{pmatrix} 0 & -iD_1^\dagger \\ iD_1 & 0 \end{pmatrix} , Q_2 = \begin{pmatrix} 0 & D_2^\dagger \\ D_2 & 0 \end{pmatrix} , \tag{20}$$

where $D_i : \mathcal{H}_+ \to \mathcal{H}_-$, $i = 1, 2$ are a couple of operators acting on $\mathcal{H}_+$ and $D_i^\dagger$ are their adjoints. Enforcing the superalgebra, namely Eqs. (10) and (11), this representation leads to the following set of compatibility conditions for D_i:

$$D_1^\dagger D_2 = D_2^\dagger D_1 , \tag{21}$$
$$D_1 D_2^\dagger = D_2 D_1^\dagger , \tag{22}$$
$$D_1^\dagger D_1 = D_2^\dagger D_2 , \tag{23}$$
$$D_1 D_1^\dagger = D_2 D_2^\dagger . \tag{24}$$

In view of Eqs. (11), (23), and (24), the Hamiltonian takes the form:

$$H = \begin{pmatrix} D_i^\dagger D_i & 0 \\ 0 & D_i D_i^\dagger \end{pmatrix} . \tag{25}$$

The latter relation together with Eq. (18) and the identities:

$$ker(D_i^\dagger D_i) = ker(D_i) , \quad ker(D_i D_i^\dagger) = ker(D_i^\dagger) , \tag{26}$$

lead to the desired result [4], namely

$$\text{index}_W = dim(ker\, D_i) - dim(ker\, D_i^\dagger) , \tag{27}$$

for either of $i = 1, 2$. In fact, Witten chooses $D_1 = D_2$ to satisfy the compatibility conditions (21)–(24). If now one identifies $\mathcal{H}_\pm$ with abstract inner product (Hilbert) spaces Γ_1 and Γ_2, and $D_i : \Gamma_1 \to \Gamma_2$ with two (parameter dependent) Fredholm operators, then Eq. (27) implies:

$$\text{index}_W = \text{index}^{\text{Analytic}}(D_i) , \tag{28}$$

for both $i = 1, 2$. In particular, one can choose Γ_a ($a = 1, 2$) to be spaces of smooth sections of a pair of complex Hermitian vector bundles E_a and D_i a pair of elliptic differential operators. Then, one has:

$$\text{index}_W = \text{index}^{\text{Atiyah-Singer}}(D_i) , \tag{29}$$

where by the Atiyah-Singer index, we mean the *topological index* introduced by Atiyah and Singer [3]. Eq. (29) is proven for twisted Dirac operators and other classical elliptic operators using the path integral techniques. The former result together with a result of K-theory lead to a proof of the general index theorem, [5, 6].

3 Parasupersymmetric Topological Invariant

In Ref. [10], a detailed analysis of both the R-S and the B-d ($p = 2$)–PSQM is presented. Here the relevant results are quoted without proof for brevity.

Consider the R-S parasuperalgebra (1), (2) written in terms of the self-adjoint generators (9):

$$Q_1^3 - \{Q_1, Q_2^2\} - Q_2 Q_1 Q_2 = 0 \tag{30}$$

$$Q_2^3 - \{Q_2, Q_1^2\} - Q_1 Q_2 Q_1 = 0 \tag{31}$$

$$[Q_1, H] = [Q_2, H] = 0 \tag{32}$$

$$Q_1^3 = 2Q_1 H \tag{33}$$

$$Q_2^3 = 2Q_2 H \quad . \tag{34}$$

These relations together with Eq. (9) which takes the form:

$$H = \frac{1}{4}\left[(Q_1^2 + Q_2^2)^2 - 3[Q_1, Q_2]^2\right]^{\frac{1}{2}} \ , \tag{35}$$

and Eqs. (14)–(16) lead to the following results:

1) The spectrum is non-negative.
2) Every parafermionic positive energy state is accompanied with a pair of parabosonic states of the same energy.[1] Here one also uses the assumption that the involution operator τ is independent of the details of the dynamics, i.e., the Hamiltonian. See [10] for more details.
3) In a basis which diagonalizes both H and Q_1, one has the following representations for Q_1, Q_2, τ, and H

$$Q_1|_{\mathcal{H}_E} \quad = \quad \sqrt{2E}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \sqrt{2E} J_3^{(1)} \ ,$$

[1]Note that here one means by the parafermionic and parabosonic states the states associated with the subspaces $\mathcal{H}_-$ and $\mathcal{H}_+$ defined by τ through Eq. (8).

$$
\begin{aligned}
Q_2|_{\mathcal{H}_E} &= \sqrt{E}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \sqrt{2E} J_1^{(1)} \, , \\
\tau|_{\mathcal{H}_E} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \, , \\
H|_{\mathcal{H}_E} &= E\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \, ,
\end{aligned}
\tag{36}
$$

where $J_i^{(1)}$ are the $(j=1)$–representations of the generators of $SU(2)$. Switching to a basis which diagonalizes τ and H, one has:

$$
\begin{aligned}
Q_1|_{\mathcal{H}_E} = \sqrt{2E}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad &, \quad Q_2|_{\mathcal{H}_E} = \sqrt{2E}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\tau|_{\mathcal{H}_E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad &, \quad H|_{\mathcal{H}_E} = E\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \, .
\end{aligned}
\tag{37}
$$

4) In view of Eq. (36), Q_1, Q_2, and H also satisfy the B-D $(p=2)$–superalgebra (3), and consequently the following simpler set of relations:

$$Q_1Q_2Q_1 = Q_2Q_1Q_1 = 0 \tag{38}$$
$$\{Q_1, Q_2^2\} = Q_1^3 \tag{39}$$
$$\{Q_2, Q_1^2\} = Q_2^3 \tag{40}$$
$$2Q_1H = Q_1^3 \tag{41}$$
$$2Q_2H = Q_2^3 \, . \tag{42}$$

Items (1) and (2) imply the topological invariance of $\Delta^{(p=2)}$ of Eq. (5). The argument is identical with the one presented for verifying topological invariance of the Witten index. Namely, under a continuous deformation of the parasupersymmetric system the energy levels may move arbitrarily but continuously. In this process some of the positive energy states may collapse to the zero level or some

zero-energy states may elevate to positive energies. However, these are only possible if the degeneracy structure is preserved. This constraint implies invariance of $\Delta^{(p=2)}$ of Eq. (5) under the deformation. The latter is also implicit in the form of the representation of τ for positive energy levels as depicted in Eq. (37).

Furthermore, item (3) may be employed to obtain an algebraic expression for the topological invariant. In order to derive such an expression, first consider the following representation of the Hilbert space:

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_+^1 \\ \mathcal{H}_+^2 \\ \mathcal{H}_- \end{pmatrix} \ , \quad \text{with} \quad \mathcal{H}_+ =: \begin{pmatrix} \mathcal{H}_+^1 \\ \mathcal{H}_+^2 \end{pmatrix} \ . \tag{43}$$

In view of the constructions (20) and Eqs. (37), we propose:

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & D_1^\dagger \\ 0 & D_1 & 0 \end{pmatrix} \ , \ Q_2 = \begin{pmatrix} 0 & 0 & D_2^\dagger \\ 0 & 0 & 0 \\ D_2 & 0 & 0 \end{pmatrix} \ , \tag{44}$$

where $D_i : \mathcal{H}_+^i \to \mathcal{H}_-$ $(i = 1, 2)$ are linear operators. Next we substitute the ansatz (44) in the parasuperalgebra (38)–(42) and Eq. (35) for the Hamiltonian. It turns out that Eqs. (38) are automatically satisfied, whereas Eqs. (39) and (40) lead to the following compatibility conditions:

$$(D_2 D_2^\dagger - D_1 D_1^\dagger) D_i = 0 \qquad (i = 1, 2) \ . \tag{45}$$

Imposing Eq. (45) on (35) leads to considerable simplifications in the form of the Hamiltonian. One finds:

$$H = \begin{pmatrix} D_2^\dagger D_2 & 0 & 0 \\ 0 & D_1^\dagger D_1 & 0 \\ 0 & 0 & D_1 D_1^\dagger + D_2 D_2^\dagger \end{pmatrix} \ . \tag{46}$$

In view of (45) and (46), Eqs (41) and (42) also are satisfied as identities.

Finally, using Eq. (46), one can easily derive the desired expression for $\Delta^{(p=2)}$:

$$\Delta^{(p=2)} = dim(ker\ D_1) + dim(ker\ D_2) - 2dim(ker\ D_1^\dagger \cap ker\ D_2^\dagger) \ . \tag{47}$$

Here we have employed the following identifications:

$$\begin{aligned} n_0^{\pi B} &= dim(ker\ D_1^\dagger D_1 \oplus ker\ D_2^\dagger D_2) \\ &= dim(ker\ D_1) + dim(ker\ D_2) \end{aligned} \tag{48}$$

$$\begin{aligned} n_0^{\pi F} &= dim(ker\ [D_1\, D_1^\dagger + D_2\, D_2^\dagger]) \\ &= dim(ker\ D_1^\dagger \cap ker\ D_2^\dagger) \ . \end{aligned} \tag{49}$$

In Eqs. (48) and (49) use is made of relations (26).

It turns out that conditions (45) may be used to simplify the expression for $\Delta^{(p=2)}$. To see this let us define $A_i := D_i D_i^\dagger$, (i=1,2). Then multiplying Eqs. (45) by $D_i^\dagger$ from the right and writing the resulting equations in terms of A_i, one has:

$$(A_1 - A_2)A_1 = 0 \ , \qquad (A_1 - A_2)A_2 = 0 \ . \tag{50}$$

In view of the fact that A_i are self-adjoint, positive definite operators, Eqs. (50) imply $ker\ A_1 = ker\ A_2$. This together with the identities (26), leads to $ker\ D_1^\dagger = ker\ D_2^\dagger$. Thus, we have:

$$\Delta^{(p=2)} = \mathrm{index}^{\mathrm{analytic}}(D_1) + \mathrm{index}^{\mathrm{analytic}}(D_2) \ . \tag{51}$$

Eq. (51) provides the desired mathematical interpretation for the parasupersymmetric topological invariant considered in this letter.

4 Conclusion

We conclude this letter by remarking that the introduction of $\Delta^{(p=2)}$ directly depended on the choice of the Hamiltonian, i.e., Eq. (35). One may try to investigate other possible forms of the Hamiltonian which are compatible with the defining parasuperalgebras of $(p = 2)$–PSQM, and even attempt to classify the corresponding systems and their topological invariants. Another possible direction of further research is to investigate the topological aspects of PSQM of orders: $p > 2$.

Acknowledgements

The auther would like to thank Drs. V. Karimipour, S. Rouhani, and A. Rezaii for invaluable comments and discussions.

References

[1] M. F. Atiyah and I. M. Singer, Bull. Amer. Math. Soc. **69**, 422 (1963); Ann. Math. **87**,546 (1968); See also [2, 3].

[2] R. S. Palais, "Seminar on the Atiyah-Singer Index Theorem," Ann. of Math. Study, Vol. 57, Princeton Uni. Press, Princeton (1965).

[3] P. Shanahan, "The Atiyah-Singer Index Theorem," Lect. Notes in Math. Vol. 638, Springer, New York (1978).

[4] E. Witten, Nucl. Phys. **B202**, 253 (1982).

[5] L. Alvarez-Gaume, Commun. Math. Phys. **90**, 161 (1983); J. Phys. A: Math. Gen. **16**, 4177 (1983); P. Windey, Acta Physica Polonica **B15**, 453 (1984); For a more complete list of references see [6].

[6] A. Mostafazadeh, J. Math. Phys. **35**, 1095 (1994).

[7] V. A. Rubakov and V. P. Spiridonov, Mod. Phys. Lett. **A3**, 1337 (1988).

[8] A. Khare, J. Phys. A: Math. Gen. **25**, L749 (1992).

[9] J. Beckers and N. Debergh, Nucl. Phys. **B340**, 767 (1990).

[10] A. Mostafazadeh, *Spectrum degeneracy of general ($p = 2$)–parasupersymmetric quantum mechanics and parasupersymmetric topological invariants,* hep-th/9410180, revised and resubmitted for publication to I. J. Mod. Phys. A (April 1995).